Protecting Entangled States Via Environment-Assisted Error Correction

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Abstract: This is a synopsis of our recent work on an environment-assisted error correction scheme of protecting entangled states coupled to a phase damping noise (see arXiv:1305.4627, Phys. Rev. A, in press). We extend the standard random unitary (RU) scheme into an error correction approach based a non-RU decomposition. We show that non-RU decomposition can be used to restore certain interesting entangled states in a dephasing channel.

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1. Introduction

In this paper, we consider an environment-assisted error correction scheme to protect a quantum state under the influence of environmental noises [1]. Classical and quantum noises are ubiquitous. The influence of the noise on a quantum system can result in disentanglement and loss of quantum coherence [2–7]. Up to now, many theoretical schemes have been proposed to control decoherence including dynamic decoupling control [8], feedback control [9], weak measurement [10] and error correction codes [11], to name a few. However, in all realistic applications, the errors in quantum information processing cannot be eliminated completely.

By extending the idea of restoring a closed system evolution to an open system, Gregoratti and Werner explicitly provided an error correction scheme which needs only one reversal operation on the system based on the outcome of measurement performed on the environment [12]. For an open quantum system, the evolution of the reduced density operator \( \rho_S \) can be written in the form of Kraus (operator sum) representation [3, 13, 14]:

\[
\rho_S(t) = \sum_n \langle n | U_0 | 0 \rangle \rho_S(0) \langle 0 | U_0^\dagger | n \rangle = \sum_n K_n \rho_S(0) K_n^\dagger,
\]

(1)

where \( U_0 = e^{-iH_{\text{tot}}t} \) (setting \( \hbar = 1 \)) is the total unitary evolution operator. Clearly, the system will collapse into the state \( K_n \rho_S(0) K_n^\dagger \), when we observe the \( n \)th outcome from the measurement on the environment. If these Kraus operators \( K_n \) are all proportional to unitary operators as \( K_n = c_n U_n \), such a decomposition is called a random unitary (RU) decomposition. Then, the initial state can be recovered by applying the reversal operation \( R_n = c_n U_n^{-1} \) depending on outcome \( n \), i.e.,

\[
\rho_R(t) = R_n K_n \rho_S(0) K_n^\dagger R_n^\dagger = \rho_S(0), \quad (\text{for all } n).
\]

(2)

This error correction scheme is of interest in many scenarios where a quantum measurement may be performed on the environment [15–20]. However, this scheme depends on the form of the Kraus decomposition, if there is an RU decomposition, then an arbitrary initial state can be fully recovered.

Generally, it is not clear whether or not the required RU decomposition exists for a quantum channel. In fact, it is known that an RU decomposition may not exist in many physical models [15, 16]. Even in the case for which the desired RU decomposition does exist, finding the RU-type Kraus operators is still a difficult task. In this paper, we consider an example of using non-RU decomposition to fully recover some particular entangled states. That is, we do not require the Kraus operators must be proportional to a set of unitary operators. This example is the first step towards a more general environment-assisted error correction scheme based on a non-RU decomposition.

2. Entanglement Restoration Based on Non-RU Decomposition

In this section, we will present a case where the standard RU scheme can be extended to a non-RU scheme if we are aiming at restoration of some particular entangled states. As an example, we consider two qubits interacting with a dephasing common bath. The Hamiltonian of this two-qubit model is \( H = \omega_0 S_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k g_k S_z (b_k + b_k^\dagger) \), where
\[ S_z = \frac{1}{2} (\sigma^A + \sigma^B) \]

Assuming the initial state of environment is the vacuum state \( |0\rangle \), in the interaction picture, the solution of the Schrödinger equation is \( |\psi_{\text{int}}(t)\rangle = U|\psi_S(0)\rangle|0\rangle \), where

\[
U = \prod_k e^{-i\phi_k(t) S_z^k} e^{-iS_b^k G_{k}(t)} e^{-iS_b^k G_{k}^*(t)},
\]

is the total evolution operator, and \( G_k(t) = \int_0^t G_k e^{i\omega_k s} ds \). The function \( \phi_k(t) \) can be determined from the equation \( \frac{d}{dt} \phi_k(t) = -i\omega_k e^{-i\phi_k(t)} G_k(t) \). Then, we can compute the inner product

\[
\langle \{m_k\}|U|0\rangle = \prod_k e^{-i\phi_k(t) S_z^k} \frac{[-iS_z G_k(t)]^{m_k}}{m_k!},
\]

where the notation \( |\{m_k\}\rangle \equiv |m_1, m_2, m_3, \cdots \rangle \equiv \otimes |m_k\rangle \) represents the multi-mode Fock state with \( m_k \) photons in the \( k \)th mode. In Eq. (4), each excitation in the environment contributes one \( S_z \) operator, so the product contains \( (S_z)^m \) operators (\( m = \sum m_k \) is the total photon number). If the total photon number \( m \) is odd, we have \( (S_z)^m = S_z \); if the total photon number \( m \) is even, we have \( (S_z)^m = S_z^2 \); if all the \( m_k \) are zero, only the first term \( e^{-i\phi_k(t) S_z^2} \) is left. Finally, we have the following three types of Kraus operators:

1. If \( |\{m_k\}\rangle = |0\rangle \), then
   \[
   \langle \{m_k\}|U|0\rangle = I - [1 - l_1(t)] S_z^2,
   \]
   where the time-dependent element is
   \[
   l_1(t) = \exp \left[ -\int_0^t dt' \int_0^{t'} \alpha(t', s) ds \right],
   \]
   with \( \alpha(t', s) = \sum_k g_k^2 e^{-i\omega_k(t'-s)} \).

2. If the total photon number \( m = \sum m_k \) in \( |\{m_k\}\rangle \) is odd, then
   \[
   \langle \{m_k\}|U|0\rangle = F_{\{m_k\}}^{\text{odd}}(t) S_z,
   \]
   where \( F_{\{m_k\}}^{\text{odd}} = \prod_k e^{-i\phi_k(t)} \frac{[-iS_z G_k(t)]^{m_k}}{m_k!} \).

3. If the total photon number \( m = \sum m_k \) in \( |\{m_k\}\rangle \) is even but not 0, then
   \[
   \langle \{m_k\}|U|0\rangle = F_{\{m_k\}}^{\text{even}}(t) S_z^2,
   \]
   where \( F_{\{m_k\}}^{\text{even}} = \prod_k e^{-i\phi_k(t)} \frac{[-iS_z G_k(t)]^{m_k}}{m_k!} \).

The reduced density matrix can be recovered by these three types of Kraus operators,

\[
\rho_S(t) = \sum_{\{m_k\}} \langle \{m_k\}|U|0\rangle \rho_S(0)|0\rangle\langle 0|\{m_k\}\rangle = \sum_{i=1}^3 K_i \rho_S(0) K_i^\dagger.
\]

In the matrix form, these non-RU-type Kraus operators \( K_i \) can be explicitly written as:

\[
K_1 = \begin{bmatrix}
  l_1(t) & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & l_1(t)
\end{bmatrix},
\]

\[
K_2 = \begin{bmatrix}
  l_2(t) & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & -l_2(t)
\end{bmatrix},
\]

\[
K_3 = \begin{bmatrix}
  l_3(t) & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & l_3(t)
\end{bmatrix}.
\]
where the time-dependent coefficients are given by $l_1(t) = \exp[-\int_0^t dt' \int_0^{t'} \alpha(t',s)ds]$, $l_2(t) = \sqrt{\sum_{\langle m_k \rangle} |F_{e_{\text{odd}}}^{m} |^2}$, and $l_3(t) = \sqrt{\sum_{\langle m_k \rangle} |F_{e_{\text{even}}}^{m} |^2}$. The corresponding measurement operators for this set of Kraus operators are $M_1 = \{|0\rangle\langle 0| \}$, $M_2 = \sum_{\langle m_k \rangle} \{|m_k\rangle\langle m_k| \}$, and $M_3 = \sum_{\langle m_k \rangle} \{|m_k\rangle\langle m_k| \}$, where $m$ is the total photon number summed over all modes. Performing a measurement with this set of measurement operators, the possible outcomes give rise to the following non-RU decomposition,

$$\text{tr}_E[U\rho(0)U^\dagger M_i] = K_i \rho_S(0) K_i^\dagger, \quad (i = 1, 2, 3)$$

with the probability $p_i = \text{tr}[K_i \rho_S(0) K_i^\dagger]$.

It is easy to see that this set of Kraus operators $K_i$ are of non-RU type. However, we can show below that one can exploit the non-RU decomposition to recover quantum entangled states. This is an interesting example showing that the restoration based on this set of non-RU Kraus operators can go beyond the standard RU scheme [12]. As an illustrative example, we consider two major types of initially entangled states $|\Phi\rangle = \alpha|00\rangle + \beta|11\rangle$ and $|\Psi\rangle = \alpha|01\rangle + \beta|10\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. Since the second one is decoherence-free in passage through the channel (9), so we only focus on the first type, $|\Phi\rangle$. For this type of initially entangled states, the restoration operators are

$$R_1 = l_1^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (14)$$

$$R_2 = l_2^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (15)$$

$$R_3 = l_3^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (16)$$

It is easy to check that

$$R_i K_i |\Phi\rangle \langle \Phi| K_i^\dagger R_i^\dagger = |\Phi\rangle \langle \Phi|, \quad (i = 1, 2, 3) \quad (17)$$

which means that the restoration operations recover the unknown initial state precisely in all possible outcome scenarios. Notably, an experimental realization can be made by measuring the environment in the Fock basis. Based on the results, $|0\rangle$ state, odd state, or even state, we can perform a corresponding restoration operation $R_1$, $R_2$, or $R_3$ respectively. The restoration procedure is explicitly shown in Fig. 1.

We note that the experimental setup for this type of restoration scheme may be implemented within the existing technologies since the parity detection [21, 22] can be realized in many practical physical systems. We expect that the non-RU decomposition will be of interest for protecting some important entangled states.

3. Discussion

A more careful examination on the decoherence process may be useful in casting a new insight into this environment-assisted error correction scheme. In the above example, measurement results on parity of the environment will project the system into three quantum trajectories. However, the resultant quantum states of the system can be categorized into the following two types conditioning on the measurement outputs:

1. If the total photon number is zero or even, the initial state of the system $|\Phi\rangle = \alpha|00\rangle + \beta|11\rangle$ become

   $$K_1 |\Phi\rangle \langle \Phi| K_1^\dagger \rightarrow |\Phi\rangle \langle \Phi|, \quad (18)$$

or

   $$K_3 |\Phi\rangle \langle \Phi| K_3^\dagger \rightarrow |\Phi\rangle \langle \Phi|, \quad (19)$$
up to a normalization factor.

(2) If the total photon number is odd, the initial state of the system $|\Phi\rangle = \alpha |00\rangle + \beta |11\rangle$ become

$$K_2 |\Phi\rangle \langle K'_2 | \rightarrow |\Phi'\rangle \langle \Phi'|,$$

(20)

up to a normalization factor, where $|\Phi'\rangle = \alpha |11\rangle - \beta |00\rangle$.

If the measurement results are not specified, then the final state of the system should be a mixture of two types of trajectories,

$$\rho(t) = P_+ |\Phi\rangle \langle \Phi| + P_- |\Phi'\rangle \langle \Phi'|$$

(21)

where $P_+$ and $P_-$ are the probabilities of obtaining $|\Phi\rangle$ and $|\Phi'\rangle$ respectively.

It should be noted that a single quantum trajectory maps pure state to pure state, but the mixture of different trajectories will cause decoherence due to the phase uncertainty. Therefore, the state of the system without specifying the measurement results must be described by a density matrix. For the total system, the quantum information should be conserved, so in principle the lost information of the system can be retrieved from the environment. By measuring the environment, it is possible to restore the information (coherence) that is originally stored in the system. Fig. 2 diagrammatically depicts an information lost-gain process.

In Fig. 2, we plot the probabilities of obtaining $|\Phi\rangle$ and $|\Phi'\rangle$, i.e., $P_+$ and $P_-$, as functions of time. We also plot the coherence evolution. The initial state $|\Phi\rangle$ gradually evolves into $|\Phi'\rangle$ with certain probability. When the probabilities of two types of trajectories are equal, the coherence is completely lost. The figure is plotted for a simple case that the environment only contains one mode. For a realistic environment, we may not see this simple periodical revival pattern. For example, in Markov case, the evolution is simply monotonic. During the time evolution, the final density matrix of the system is always a mixture of $|\Phi\rangle \langle \Phi|$ and $|\Phi'\rangle \langle \Phi'|$ with certain probabilities $P_+(t)$ and $P_-(t)$, respectively. Given the time evolution of $P_+(t)$ and $P_-(t)$ in Fig. 2, it is easy to understand that when $P_+(t) = P_-(t) = 0.5$ at a certain time point, the density matrix become

$$\rho(t) = P_+ |\Phi\rangle \langle \Phi| + P_- |\Phi'\rangle \langle \Phi'|$$

(22)

which is a diagonal matrix without coherence terms (off-diagonal elements) as Fig. 3 shows.
This example has shown that although the quantum map $\rho_S(0) \rightarrow \rho_S(t)$ can not be decomposed into random unitary trajectories for an arbitrary initial state, it is still possible to find some particular initial states evolving in a subspace, in which the decomposition keeps the RU form. More specifically, we see that the initial states in the form of $|\Phi\rangle = \alpha|00\rangle + \beta|11\rangle$ evolve in the sub-space $\{|00\rangle, |11\rangle\}$, and all the three Kraus operators are RU form in this subspace. Therefore, it is still possible to use the inverse operations to recover the initial state from the trajectories in each category. Our new scheme relaxes the limitation that the quantum map must be decomposed into a RU form. This implies the environment-assisted error correction schemes may be implemented to protect some interesting initial states rather than the general quantum channel.

4. Conclusion

We have shown how to use the environment-assisted error correction scheme to eliminate quantum errors caused by a dephasing channel. The proposed error correction scheme can be accomplished with one projective measurement on the environment and one unitary reversal operation on the system. In principle, the restoration procedure is deterministic, not probabilistic, and ideally the successful rate is 1. As an concrete realization, we showed that the required measurement on the environment can be performed in the Fock basis, requiring only the parity of the photon numbers in all the participating modes.

Moreover, we went beyond the original restoration scheme proposed in [12], which is based on RU decomposition. We showed that some non-RU decompositions can also be used to recover certain important entangled states. Our research provides a possibility of finding more environment-assisted error correction schemes based on a non-RU decomposition of quantum channels.

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References